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Axisymmetric Hypersonic Flow with Mass Transfer and Large Transverse Curvature

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A theory is presented for the combined effects of mass transfer and transverse curvature on the hypersonic flow of a perfect gas in the limit where an axisymmetric body is vanishingly thin compared with an adjacent, single-species laminar boundary layer. Arbitrary distributions of body temperature, mass transfer, and thickness are considered. The major boundary-layer properties are presented as analytic functions of Prandtl number, specific heat ratio, exponent of the power-law viscosity relation, body enthalpy ratio, a mass-transfer parameter, and a transverse-curvature parameter. The expressions for skin-friction, drag, and heat-transfer estimates are valid for any shock-wave strength but require knowledge of the order of magnitude of surface pressure. This is provided for the limiting cases of very strong and very weak shock waves. Numerical results are also presented for self-similar hypersonic flow over slender $\frac{3}{4}$ power-law bodies with mass transfer and transverse curvature effects ranging from small to large. The present theory is compared with these numerical solutions and, although exact only in the limit of large transverse curvature, is found to predict accurately the relative effects of mass transfer.

Nomenclature

A = transverse curvature parameter, Eq. (2.5)
E = mass-transfer parameter, Eq. (2.10)
$\bar{f},\bar{g}$ = dependent variables defined in (3.11) and (3.12)
f,g = dependent variables defined in (2.1b)
H = stagnation enthalpy, $h + u^2/2$
h = enthalpy
J = see Eqs. (3.20) and (3.21)
j = see Eq. (3.31)
L,r_b = body length and base radius
M = Mach number
Q^* = effective heat of ablation, $(q/\rho v)_w$
R = r^2/r_w^2
r_e = effective body ordinate
S = see Eq. (3.29)
x = distance measured axially from the nose
β = pressure gradient parameter, Eq. (2.6)
δ = local boundary-layer thickness
ϵ = scale of $1 - f_\eta$ and $1 - g$ in the viscous convective layer, Eqs. (3.11) and (3.19)
ζ = convective-layer independent variable, Eq. (3.9)
η,ξ = Lees-Dorodnitsyn variables, Eq. (2.1a)
λ = $g_{\eta,w}/f_{\eta\eta,w}$
ω = viscosity exponent, $\mu \propto h^\omega$

Subscripts

e = edge of boundary layer
s = freestream stagnation condition
∞ = freestream

I. Introduction

THIS paper presents a theory for axisymmetric hypersonic flow over slender bodies with surface mass transfer, applicable when the local transverse body radius r_w is much smaller than the local boundary-layer thickness δ (i.e., limit of large transverse curvature). The situation $r_w \ll \delta$ arises both near the tip of a slender body when $r_w \sim x^n$ and $n > \frac{3}{4}$, and far downstream from the nose when $n < \frac{1}{2}$. We assume that the inviscid flow is defined by hypersonic small-disturbance theory and the viscous flow by conventional boundary-layer theory.

A theory for slender bodies with large transverse curvature in axisymmetric hypersonic flow with a strong shock wave was first described by Stewartson.[1] Stewartson treated a model gas, where the Prandtl number is unity and viscosity varies linearly with temperature, and an isothermal surface without mass transfer. His theory has been confirmed by Solomon.[2] The present authors have recently extended the theory by relaxing assumptions for the gas so as to describe perfect gases of arbitrary Prandtl number and power-law variation of viscosity with temperature.[3] The body remained isothermal without surface mass transfer. Bound-

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ary-layer properties in the leading approximation were obtained in closed form.‡

The only study of axisymmetric hypersonic flow with both large transverse curvature and mass transfer is that by Li and Gross.[4,5] These authors define[4] a problem for the boundary layer with binary diffusion and linear viscosity variation where the flow is self-similar, that is, where r_w varies as $x^{3/4}$ and $(\rho v)_w$ varies as $x^{-3/4}$. They point out that a solution of Stewartson type may be possible, but they do not solve the equations.

In the present paper we extend the theory for axisymmetric hypersonic flow with large transverse curvature[3] by relaxing boundary conditions at the body surface to allow streamwise variation in surface mass transfer and temperature. We consider only the case where both the primary gas from upstream and any secondary gas from the body are identical and perfect, with constant specific heat ratio γ, constant Prandtl number Pr, and power-law viscosity μ. The major boundary-layer properties are obtained analytically. The results for skin friction, drag, and heat-transfer rate are valid for any shock-wave strength. All properties are compared with numerical solutions of the self-similar, axisymmetric boundary-layer equations [for $r_w \propto x^{3/4}$, $(\rho v)_w \propto x^{-3/4}$, and a strong shock wave].

II. General Boundary-Layer Equations

The hypersonic boundary-layer equations with transverse curvature are simplified by transformation of the cylindrical coordinates (x,r) to (ξ,η);

$$\xi = \int_0^x (\rho\mu)_w u_e r_w^2 dx, \qquad \eta = u_e(2\xi)^{-1/2}\int_{r_w}^r \rho r dr \quad (2.1a)$$

Let the dependent variables be

$$f = f_w + \int_0^\eta \frac{u}{u_e} d\eta, \qquad g = \frac{H}{H_e}, \qquad R = \frac{r^2}{r_w^2} \quad (2.1b)$$

The equations for streamwise momentum, energy, and continuity are then

$$0 = [(g - mf_\eta^2)^{\omega-1} g_w^{1-\omega} R f_{\eta\eta}]_\eta + ff_{\eta\eta} + \beta(g - f_\eta^2) - 2\xi(f_\eta f_{\xi\eta} - f_\xi f_{\eta\eta}) \quad (2.2)$$

$$0 = \{(g - mf_\eta^2)^{\omega-1} g_w^{1-\omega} R[g_\eta - 2(1 - Pr)mf_\eta f_{\eta\eta}]\}_\eta + Prfg_\eta - 2Pr\xi(f_\eta g_\xi - f_\xi g_\eta) \quad (2.3)$$

$$\rho v r = -\left(\frac{d\xi}{dx}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}\right)[(2\xi)^{1/2} f] \quad (2.4)$$

where for slender bodies ($u_e \approx u_\infty$),

$$R = 1 + A\int_0^\eta (g - mf_\eta^2)d\eta, \; A = \frac{\gamma - 1}{m\gamma}\frac{u_\infty(2\xi)^{1/2}}{p_w r_w^2} \quad (2.5)$$

$$m = 1 - \frac{h_e}{H_e}, \; \beta = -\frac{\gamma - 1}{m\gamma}\frac{d\log p_w}{d\log\xi} \quad (2.6)$$

From (2.5), A is a measure of the transverse curvature δ/r_w.

We now restrict the analysis to hypersonic flows where $M_e^{-2} \ll 1$, and we set $m = 1$ in (2.2–2.6). With the additional assumption that vorticity in the inviscid layer can be neglected relative to its values in the boundary layer, the boundary conditions as $\eta \to \infty$ become

$$f_\eta \to 1, \qquad g \to 1, \qquad R_\eta \to 0 \quad (2.7)$$

At the body surface it is assumed that slip and temperature jump can be neglected and that the surface is either porous or ablating. Then at $\eta = 0$,

$$f_\eta = 0, \qquad f = f_w(\xi), \qquad g = g_w(\xi) \quad (2.8)$$

$$(\rho v r)_w = -(d\xi/dx)(2\xi)^{-1/2}(1 + 2\xi d/d\xi)f_w \quad (2.9)$$

If the surface is porous, $(\rho v)_w$ is considered known. If the surface is ablating, the ratio of the surface heat flux q_w to $(\rho v)_w$ is assumed to be known and is designated Q^* (effective specific heat of ablation). Let us now introduce a mass transfer parameter of order one:

$$E \equiv \left(\frac{\rho v}{q}\right)_w \frac{H_\infty}{Pr} = \frac{H_\infty}{Q^* Pr} \quad (2.10)$$

If the surface is ablating, E is calculated from (2.10). If $(\rho v)_w$ is specified, then E is found implicitly from

$$(\rho v)_w H_\infty = PrEq_w \quad (2.11)$$

where q_w is a function of E to be determined. Note that E is negative for wall suction and can be expressed in terms of the transformed variables as

$$E = -(f_w + 2\xi f_{\xi,w})/g_{\eta,w} \quad (2.12)$$

In the special case of an insulated porous wall, $q_w = g_{\eta,w} = 0$, $E = \infty$, and $Eq_w = O(1)$, from (2.11).

The skin-friction coefficient in transformed variables is

$$r_w C_f \equiv \frac{(r\mu\partial u/\partial r)_w}{(1/2)(\rho u^2)_\infty} = 4\frac{\mu_s}{\rho_\infty u_\infty}\frac{f_{\eta\eta,w}}{g_w^{1-\omega}A} \quad (2.13)$$

where μ_s is the viscosity evaluated at the freestream stagnation temperature: $\mu_s = \mu_\infty[\frac{1}{2}(\gamma - 1)M_\infty^2]^\omega$. The friction-drag coefficient referred to base area πr_b^2 is

$$C_{D_f} = 8\left(\frac{L}{r_b}\right)^2 \frac{\mu_s}{\rho_\infty u_\infty L}\int_0^1 \frac{f_{\eta\eta,w}}{g_w^{1-\omega}A} d\left(\frac{x}{L}\right) \quad (2.14)$$

where L is the body length. The heat-transfer rate to the wall is

$$q_w \equiv (\mu\partial H/\partial r)_w/Pr = (4Pr)^{-1}(\rho u^3)_\infty \lambda C_f \quad (2.15)$$

where $\lambda \equiv g_{\eta,w}/f_{\eta\eta,w}$.

III. Analytic Solution for $A \gg 1$

We now consider the limit of large transverse curvature by letting $A \to \infty$ uniformly in ξ. In this limit, the boundary layer can be divided into two viscous layers (Fig. 1). One is an outer, convective viscous layer in which velocity and stagnation enthalpy depart from corresponding values in the freestream by only small amounts. For an adequate description of properties in this layer, all terms must be retained in the boundary-layer equations. The other viscous layer is an inner layer adjacent to the body. In this layer, where the velocity and stagnation enthalpy profiles are principally developed, convection and pressure effects are unimportant compared with viscous diffusion.

A. Inner Viscous Layer

The need to retain or omit convective and pressure terms can be determined by ordering terms in, say, the momentum equation (2.2). Throughout the boundary layer f_η and g are of order A^0, and from (2.5), R is of order $A\eta$, $\eta \neq 0$. Then in the right member of (2.2) the first term (viscous term) is of order A/η. The terms $ff_{\eta\eta} + 2\xi f_\xi f_{\eta\eta}$ can be regrouped as $f_{\eta\eta}(f_w + 2\xi f_{\xi,w})$ and $f_{\eta\eta}(1 + 2\xi\partial/\partial\xi)(f - f_w)$, which are of order f_w/η and A^0, respectively. The remaining terms in (2.2) are of order A^0. From (2.12) f_w is required to be of order $g_{\eta,w}$, and it will be shown that this is of order A, within a factor of $\ln A$. Thus terms in (2.2) are either of order A/η or of order A^0. Clearly, in the outer convective viscous layer, η, f, $f - f_w$, and $R^{1/2}$ are of order A, and all

‡ The authors' theory has been confirmed independently by Bush (to be published).

terms are needed. In the inner viscous layer, η, $f - f_w$, and $R^{1/2}$ are much smaller than A, and terms of order A/η dominate, reducing (2.2) and (2.3) to

$$Eg_{\eta,w} f_{\eta\eta} = [(g - f_\eta^2)^{\omega-1} g_w^{1-\omega} R f_{\eta\eta}]_\eta$$

$$PrEg_{\eta,w}\, g_\eta = \{(g - f_\eta^2)^{\omega-1} g_w^{1-\omega} R [g - (1 - Pr) f_\eta^2]_\eta\}_\eta$$

which can be integrated. The result using (2.8) is

$$f_{\eta\eta,w}(1 + \lambda E f_\eta) = (g - f_\eta^2)^{\omega-1} g_w^{1-\omega} R f_{\eta\eta} \qquad (3.1)$$

$$g_{\eta,w}[1 + PrE(g - g_w)] = (g - f_\eta^2)^{\omega-1} g_w^{1-\omega} R \times [g - (1 - Pr) f_\eta^2]_\eta \qquad (3.2)$$

Elimination of R and another integration with respect to f_η provides an enthalpy integral

$$g(f_\eta) = g_w - \frac{1}{PrE}[1 - (1 + \lambda E f_\eta)^{Pr}] + \frac{2(1 + \lambda E f_\eta)}{(2 - Pr)\lambda^2 E^2}[(1 + \lambda E f_\eta)^{Pr-1} + (1 - Pr)\lambda E f_\eta - 1] \qquad (3.3)$$

When $Pr = 1$, $g - g_w = \lambda f_\eta$ for all E. The grouping $1 + \lambda E f_\eta$ that appears in (3.3) is seen from (3.1) to represent the ratio of local $R\tau$ to $(R\tau)_w$, where τ is the shear stress. Without mass transfer $R\tau$ is a constant, and the sublayer solution reduces to that for Couette flow along concentric cylinders. Since $R\tau$ is bounded, in general, (3.1) indicates that $1 + \lambda E \to 0^+$ as $\tau_w \to \infty$ (strong suction). It is later shown, (3.18), that this requires $-1 < Pr(1 - g_w)E$.

The peak temperature in the boundary layer is of interest for estimates of real gas effects. Whenever $\lambda > 0$

$$\left(\frac{h}{H_e}\right)_{\max} = g_w + \frac{1}{\lambda^2 E^2}\left\{-1 - \frac{\lambda^2 E}{Pr} + \left[1 + \frac{(2 - Pr)\lambda^2 E}{2Pr}\right]^{2/(2-Pr)}\right\} \qquad (3.4)$$

where λ is expressed as a function of g_w, E, and Pr in (3.18). When $Pr = 1$, $(h/H_e)_{\max} = g_w + \lambda^2/4$ for all E. Values of $(h/H_e)_{\max}$ for $Pr = 0.7$ were found to vary less than 4% for $E < 10$.

Equations (2.5) and (3.1) relate R, η, and f_η;

$$\ln R = \frac{g_w^{1-\omega} A}{f_{\eta\eta,w}} \int_0^{f_\eta} \frac{[g(t) - t^2]^\omega}{1 + \lambda E t}\, dt \qquad (3.5)$$

$$\eta = \frac{g_w^{1-\omega}}{f_{\eta\eta,w}} \int_0^{f_\eta} \frac{R[g(t) - t^2]^{\omega-1}}{1 + \lambda E t}\, dt \qquad (3.6)$$

The parameters sought are a constant of order A^0 (either λ or g_w) and $f_{\eta\eta,w}$. These will be evaluated by matching the flow in the inner viscous layer with that in the viscous convective layer.

B. Viscous Convective Layer

In the viscous convective layer, terms representing the effects of viscosity no longer dominate the equations of motion. With all terms retained and with $m = 1$ (2.2) and (2.3) become, after integration with (2.8),

$$(1 + \lambda E f_\eta) f_{\eta\eta,w} = (g - f_\eta^2)^{\omega-1} g_w^{1-\omega} R f_{\eta\eta} + \beta \int_0^\eta (g - f_\eta^2) d\eta + (f_\eta - 1)\left(1 + \frac{2\xi\partial}{\partial\xi}\right) \times (f - f_w) + \left(1 + \frac{2\xi\partial}{\partial\xi}\right)\int_0^\eta f_\eta(1 - f_\eta) d\eta \qquad (3.7)$$

$$[1 + PrE(g - g_w)] g_{\eta,w} = (g - f_\eta^2)^{\omega-1} g_w^{1-\omega} \times R[g - (1 - Pr) f_\eta^2]_\eta + Pr\left[(g - 1)\left(1 + \frac{2\xi\partial}{\partial\xi}\right) \times (f - f_w) + \left(1 + \frac{2\xi\partial}{\partial\xi}\right)\int_0^\eta f_\eta (1 - g) d\eta\right] \qquad (3.8)$$

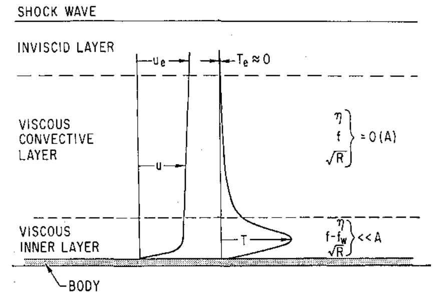


Fig. 1 Sketch of flow geometry and typical velocity and temperature profiles for hypersonic flow over axisymmetric bodies with large transverse curvature.

If new variables are properly chosen for the viscous convective layer, Eqs. (3.7, 3.8, and 2.5), when written in terms of these new variables (which are of order one in this layer), will contain neither A nor $\ln A$. Let

$$\eta = \zeta A \epsilon^\omega,\ \xi = \xi \qquad (3.9)$$

$$R = \bar{R}(\zeta,\xi) A^2 \epsilon^{\omega+1}[1 + O(\epsilon)] \qquad (3.10)$$

$$f_\eta = 1 - \epsilon \bar{f}_\zeta(\zeta,\xi) + O(\epsilon^2),\ g = 1 - \epsilon \bar{g}(\zeta,\xi) + O(\epsilon^2) \qquad (3.11)$$

where $\epsilon(A)$ vanishes as $A \to \infty$. ϵ cannot be chosen until properties of the inner and convective viscous layers are matched. (It will be found that ϵ is $\ln A$ raised to a negative power.) Assume without loss of generality that $\bar{f}(\infty,\xi) = 0$. Then

$$\partial/\partial\eta = (\zeta/\eta)\partial/\partial\zeta$$

$$\xi\partial/\partial\xi = \xi\partial/\partial\xi + (\psi - \tfrac{1}{2})\zeta\partial/\partial\zeta$$

where derivatives of ϵ and $\ln A$ have been neglected as of higher order and where

$$\psi \equiv d\ln(p_w r_w^2)/d\ln\xi \qquad (3.12)$$

Parameters (i.e., functions only of ξ) are

$$\lambda = \lambda_0 + O(\epsilon), \qquad f_{\eta\eta,w} = f_0 A^{\omega+1}[1 + O(\epsilon)] \qquad (3.13)$$

$$\int_0^\infty f_\eta(1 - f_\eta) d\eta = f_1 A \epsilon^{\omega+1}[1 + O(\epsilon)] \qquad (3.14a)$$

$$\int_0^\infty f_\eta(1 - g) d\eta = g_1 A \epsilon^{\omega+1}[1 + O(\epsilon)] \qquad (3.14b)$$

$$\int_0^\infty (g - f_\eta^2) d\eta = r_1 A \epsilon^{\omega+1}[1 + O(\epsilon)] \qquad (3.14c)$$

With these replacements, (3.7, 3.8, and 2.5) become, to leading order,

$$(1 + \lambda_0 E) f_0 + (2\bar{f}_\zeta - \bar{g})^{\omega-1} g_w^{1-\omega} \bar{R} \bar{f}_{\zeta\zeta} = \beta\bar{R} + 2(f_1 + \bar{f} - \zeta\bar{f}_\zeta)(1 - \psi) + 2\xi(f_1 + \bar{f})_\xi \qquad (3.15a)$$

$$f_0\lambda_0[1 + PrE(1 - g_w)] + (2\bar{f}_\zeta - \bar{g})^{\omega-1} g_w^{1-\omega} \bar{R} \times [\bar{g}_\zeta - 2(1 - Pr)\bar{f}_{\zeta\zeta}] = 2Pr\left(g_1 + \int_\infty^\zeta \bar{g} d\zeta - \zeta\bar{g}\right) \times (1 - \psi) + 2Pr\xi\left(g_1 + \int_\infty^\zeta \bar{g} d\zeta\right)_\xi \qquad (3.15b)$$

$$\bar{R} = r_1 + 2\bar{f} - \int_\infty^\zeta \bar{g} d\zeta \qquad (3.15c)$$

An alternate equation, involving only $\bar{R}$, can be obtained. Multiply (3.15a) by $2Pr$ and subtract (3.15b);

$$f_0[2Pr - \lambda_0 + \lambda_0 PrE(1 + g_w)] + g_w^{1-\omega}(\bar{R}_\zeta)^{\omega-1}\bar{R}\bar{R}_{\zeta\zeta} = 2Pr[\beta\bar{R} + (\bar{R} - \zeta\bar{R}_\zeta)(1 - \psi) + (1 + \xi d/d\xi)(2f_1 - g_1 - r_1) + \xi\bar{R}_\xi] \qquad (3.16)$$

The parameters of (3.13) and (3.14) will be evaluated by matching the two viscous layers in their region of overlap,

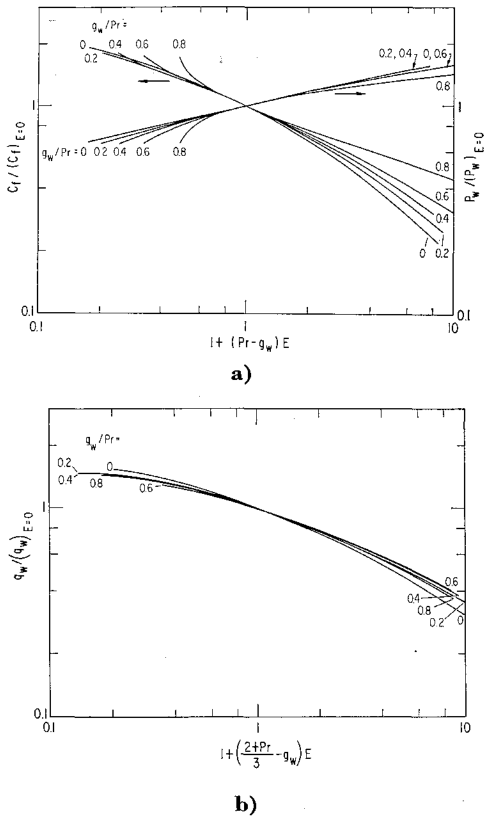


Fig. 2 Effects of mass transfer on slender axisymmetric bodies with large transverse curvature in hypersonic flow. $Pr = 0.7$, $\omega = 1$; a) shear and pressure (latter for strong shock wave); b) heat-transfer rate.

and later by application of the boundary conditions at the outer edge of the boundary layer.

C. Matching of the Two Viscous Layers

At the inner edge of the viscous convective layer R must vanish, from the scaling of (3.9) and (3.10), and the right members of (3.15a, 3.15b, and 3.16) must vanish in order to match with the inner layer. Thus from (3.15a) and (3.15b)

$$0 = f_1(\xi) + \bar{f}(0,\xi), \qquad 0 = g_1(\xi) + \int_\infty^0 \bar{g} d\zeta \quad (3.17a)$$

and from (3.16)

$$0 = \bar{R}(0,\xi), \qquad 0 = 2f_1(\xi) - g_1(\xi) - r_1(\xi) \quad (3.17b)$$

These results show that the only contribution to the integrals in (3.14) comes from the viscous convective layer.

The inner viscous-layer solution (3.3) for $g(f_\eta)$, written in terms of barred variables, matches (3.11) to order (ϵ^0)§ if λ_0 and E are related parametrically by $\lambda_0 = (L-1)/E$ and

$$E = \frac{(2-Pr)(L-1)^2(L^{Pr}-1)}{Pr[-2(L^{Pr}-1) + Pr(L^2-1) - g_w(2-Pr)(L-1)^2]} \quad (3.18)$$

where L is a dummy parameter. If Pr, g_w, and E are constant, λ_0 is a constant.

When $Pr = 1$, $\lambda_0 = 1 - g_w$ for all E. When $L \to 0^+$ (strong suction), $\lambda_0 \to Pr(1 - g_w)$. When $L \to \infty$ (strong blowing), $\lambda_0 = O(E^{(1-Pr)/Pr})$. For an insulated porous wall $\lambda_0 = 0$, and the mass-transfer parameter is the product $\lambda_0 E$, which is $O(1)$ and will be related to $(\rho v r)_w$ in (3.26). The insulated-wall enthalpy ratio is found by causing the denominator of the right member of (3.18) to vanish.

§ Matching to leading order essentially amounts to recognizing that in the inner viscous layer, $f_\eta \to g \to 1$ as $\eta \to O(A\epsilon^\omega)$ and that in the viscous convective layer, initial conditions on $\bar{R}(\zeta)$ are $\bar{R}(0) = 0$ and $(\bar{R}_\zeta)^{\omega+1} \propto |\ln\bar{R}|$, from (3.16) and (3.17). Stewartson[1] found $\bar{R}(\zeta)$, and we know $\bar{f}_\zeta$ and $\bar{g}$ similarly exist, although both are unbounded as $\zeta \to 0$, being proportional to $\bar{R}_\zeta$.

Similarly, expansion of (3.5) in terms of $\bar{R}$ and f_0 to leading order provides

$$\epsilon = (\ln A)^{-1/(\omega+1)}, \qquad f_0 = Jg_w^{1-\omega}/2 \quad (3.19)$$

where

$$J \equiv \int_0^1 \frac{[g(t) - t^2]^\omega}{(1 + \lambda_0 E t)} dt \quad (3.20)$$

and $g(t)$ is given by (3.3). Special values of J are

$$J = \begin{cases} \left.\begin{aligned} &\frac{g_w}{\lambda_0 E} - \frac{1}{Pr\lambda_0 E^2} - \frac{1}{\lambda_0^3 E^3}\ln(1+\lambda_0 E) \\ &- \frac{g_w}{Pr\lambda_0 E} + \frac{1}{2\lambda_0 E} + \frac{1}{\lambda_0^2 E^2} \end{aligned}\right\} & \omega = 1 \\ \displaystyle\int_0^1 \frac{(g_w + t)^\omega (1-t)^\omega}{[1 + (1-g_w)Et]dt'} & Pr = 1 \end{cases} \quad (3.21)$$

For an insulated porous wall, J is $O(1)$. Parameters in (3.18–3.21) are independent of the profiles in the viscous convective layer.

D. Properties at the Wall

If field properties within the viscous convective layer are required, it is necessary to integrate the partial differential equations (3.15) numerically. If, however, one is content with a solution for the inner viscous layer and surface properties, one need only collect formulae already obtained. Combining (2.13, 3.13, and 3.19), we find that the skin friction is

$$r_w C_f = (2J/\ln A)(\mu_s/\rho_\infty u_\infty) \quad (3.22)$$

$C_f/(C_f)_{E=0}$ is plotted in Fig. 2a for $\omega = 1$ and $Pr = 0.7$. Note that $C_f \sim 1/r_w$ (to within a factor $\ln A$) whenever g_w and E are constant. In these cases, the friction drag per unit length is constant. In general,

$$C_D = C_{D_f} = \frac{4}{\ln A}\left(\frac{L}{r_b}\right)^2 \frac{\mu_s}{\rho_\infty u_\infty L} \int_0^1 J d\left(\frac{x}{L}\right) \quad (3.23)$$

The drag coefficient is due to friction, since the order of magnitude of the pressure-drag coefficient never exceeds the square root of that of the friction-drag coefficient, which is found to be large compared with one. Similarly, the heat-transfer rate is

$$r_w q_w = (\mu_s H_\infty/Pr)(\lambda_0 J/\ln A) \quad (3.24)$$

The ratio $q_w/(q_w)_{E=0}$ is plotted in Fig. 2b for $\omega = 1$ and $Pr = 0.7$. To complete the evaluation of C_f and q_w we need an expression for $\ln A$ valid to leading order. Since $\ln A \gg 1$, only the order of magnitude of A is required, the resulting error being of higher order than terms of order $\epsilon = (\ln A)^{-1/(\omega+1)}$ neglected in the viscous convective layer. From (2.1) and (2.6)

$$A^2 = O\left(\frac{x^2}{r_w^2}\frac{M_\infty^{2+2\omega}\nu_\infty}{g_w^{1-\omega}u_\infty x}\frac{p_\infty}{p_w}\right)$$

If the pressure ratio p_w/p_∞ is very large compared with one, p_w/p_∞ is known[3] to be of order $M_\infty^{2+\omega}(\nu_\infty/u_\infty x)^{1/2}$, and in the linearized limit $p_w \approx p_\infty$. Then

$$\ln A = \begin{cases} \dfrac{1}{2}\ln\left[\dfrac{x^2}{r_w^2}\dfrac{M_\infty^\omega}{g_w^{1-\omega}}\left(\dfrac{\nu_\infty}{u_\infty x}\right)^{1/2}\right], & \dfrac{p_w}{p_\infty} \gg 1 \quad (3.25a) \\ \ln\left[\dfrac{x}{r_w}\dfrac{M_\infty^{1+\omega}}{g_w^{(1-\omega)/2}}\left(\dfrac{\nu_\infty}{u_\infty x}\right)^{1/2}\right], & \dfrac{p_w}{p_\infty} - 1 \ll 1 \quad (3.25b) \end{cases}$$

Equations (3.22–3.24) depend on the mass-transfer parameter

E, which is to be evaluated directly from (2.10) for an ablating body. For a porous body where $(\rho v r)_w$ is given, the corresponding value of E is found implicitly from

$$[(\rho v r)_w/\mu_s] \ln A = \lambda_0 E J \quad (3.26)$$

Figure 3a presents $[(\rho v r)_w/\mu_s] \ln A$ as a function of E for $Pr = 0.7$ and $\omega = \frac{3}{4}$.

The effect of mass injection ($E > 0$) on the surface flow properties is to reduce both C_f and q_w, as shown by the following ratios for $|E| \ll 1$ and $\omega = 1$:

$$\frac{C_f}{(C_f)_{E=0}} = \frac{J}{(J)_{E=0}} \sim 1 - \frac{[Pr(1-g_w) + 3g_w](Pr - g_w)E}{2(Pr + 3g_w)}$$

$$\frac{q_w}{(q_w)_{E=0}} = \frac{\lambda_0 J}{(\lambda_0 J)_{E=0}} \sim 1 - \frac{Pr[Pr(2+Pr) + 3g_w(2 - Pr - 2g_w)]E}{6(Pr + 3g_w)}$$

where

$$(J)_{E=0} = (Pr + 3g_w)/6, \ (\lambda_0)_{E=0} = Pr - g_w$$

$$E \sim \begin{cases} H_\infty/(PrQ^*), & \text{ablating body} \\ \dfrac{(\rho v r)_w}{\mu_s} \dfrac{6 \ln A}{(Pr + 3g_w)(Pr - g_w)}, & \text{porous body} \end{cases}$$

The present results indicate the correction to apply to $\omega = 1$ solutions for shear, heat transfer, and displacement to account for $\omega \neq 1$. In the $\omega = 1$ solution $\nu_\infty/u_\infty x$ is replaced by $C\nu_\infty/u_\infty x$, where $C = (\mu_r/\mu_\infty)(T_\infty/T_r)$ and T_r is a reference temperature defined by $T_r/T_s = [J/(J)_{\omega=1}]^{1/(\omega-1)}$. The quantity $\mu_s/(\rho v r)_w$ is left unchanged in (3.26). $T_r/(T_r)_{E=0}$ is plotted for $\omega = \frac{3}{4}$ and $Pr = 0.7$ in Fig. 3b. Since C varies as $T_r^{\omega-1}$, there is little change in C with E for $E < 9$. If $E \gg 1$, the variation of C may be significant.

E. Location of the Boundary-Layer Edge

As $\zeta \to \infty$, $\zeta \bar{R}_\zeta$, $\zeta \bar{f}_\zeta$, and $\zeta \bar{g}$ vanish algebraically if $\omega < 1$ and exponentially if $\omega = 1$ (as noted for plane flow by Freeman and Lam[6] in Crocco variables and by Bush[7] in von Mises variables). Equation (3.16) becomes, as $\zeta \to \infty$,

$$f_0[2Pr - \lambda_0 + \lambda_0 PrE(1 + g_w)]/(2Pr) = (1 - \psi + \beta)\bar{R}(\infty,\xi) + \xi \bar{R}_\xi(\infty,\xi) \quad (3.27)$$

Equation (3.27) is the momentum integral equation times $2Pr$, minus the energy integral equation, expressed in viscous convective layer variables. Use of expressions (2.6, 3.10, and 3.12) and the integrating factor $\xi r_w^{-2} p_w^{-(2\gamma-1)/\gamma}$ allows (3.27) to be integrated as a relation between $R(\infty,\xi)$ and p_w:

$$r_w^2 R(\infty,\xi) = \frac{\gamma - 1}{\gamma} \frac{\mu_s u_\infty}{\ln A} p_w^{-1/\gamma} \int_0^x JS^2 p_w^{-(\gamma-1)/\gamma} dx \quad (3.28)$$

where

$$S^2 \equiv 2 - (\lambda_0/Pr) + \lambda_0 E(1 + g_w) \quad (3.29)$$

When $Pr = 1$, $S^2 = 1 + g_w + (1 - g_w^2)E$. Equation (3.28), like those for C_f and q_w already obtained, becomes exact as $\delta/r_w \to \infty$ for any shock strength and any continuous $g_w(\xi)$, $E(\xi)$.

The effective body ordinate r_e for an equivalent inviscid flow for $h_e/H_e \neq 0$ is defined by[8]

$$\int_{r_w}^{r_e} \rho_e u_e r dr = \int_{r_w}^{\infty} (\rho_e u_e - \rho u) r dr + \int_0^x (\rho v r)_w dx$$

In the hypersonic limit $h_e/H_e \to 0$, assuming f_w is of order $A/\ln A$, r_e is given by[4,5]

$$(r_e/r_w)^2 = R(\infty,\xi) = (1 + \delta/r_w)^2 \approx (\delta/r_w)^2 \quad (3.30)$$

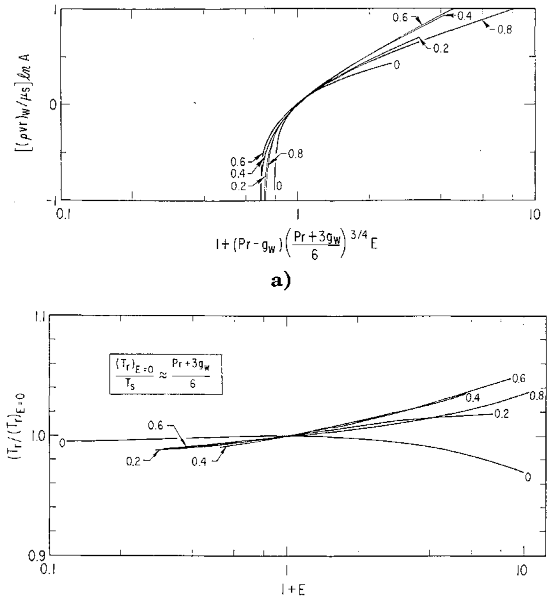


Fig. 3 Effects of mass transfer on slender axisymmetric bodies with large transverse curvature in hypersonic flow. $Pr = 0.7$, $\omega = \frac{3}{4}$; a) relation between $(\rho v)_w$ and E; b) reference temperature for linear viscosity approximation.

which is the same expression as that for a hypersonic boundary layer without blowing.

F. Boundary-Layer Pressure and Displacement

To obtain explicit expressions for boundary-layer pressure and displacement thickness, one needs another relation between them to supplement (3.28) and (3.30). Such a relation is obtained from the inviscid layer (Fig. 1). Cases where $p_w/p_\infty \gg 1$ (strong-shock limit) or $p_w/p_\infty - 1 \ll 1$ (linearized limit) are treated below.

1. *Strong-shock limit*

The situation where both δ/r_w and p_w/p_∞ are large may arise near the nose of a slender body where $r_w \sim x^n$ and $n > \frac{3}{4}$ (or when $n = \frac{3}{4}$, $A \to \infty$). A similarity solution for the inviscid layer is possible provided g_w and E are constant and the shock is strong.

In this special case, r_e will be found to vary as $(x^3/\ln A)^{1/4}$. The inviscid flow equations of leading order have the same form and solution as those for constant A (which correspond to known, self-similar flow over a $\frac{3}{4}$ power-law body) with an error of order $(\log A)^{-1}$. Numerical results[9,10] lead t.

$$p_w/p_\infty = (jM_\infty r_e/x)^2 \quad (3.31)$$

where $j = 0.685, 0.846, 0.941$ for $\gamma = 1, 1.4, 1.67$.

Combining (3.28, 3.30, and 3.31) leads to

$$\frac{\delta^4}{x^4} = \frac{16\gamma JS^2}{j^2(3\gamma - 1)} \left(\frac{\gamma - 1}{2}\right)^{\omega+1} \frac{(M_\infty^{2\omega}\nu_\infty/u_\infty x)}{\ln(x^3 M_\infty^{2\omega}\nu_\infty/u_\infty r_w^4)} \quad (3.32)$$

(which shows that $r_e^4 \sim x^3/\ln A$ as assumed). The shock-wave ordinate r_* is then $r_*/\delta = 1, 1.143, 1.221$ for $\gamma = 1, 1.4, 1.67$, and p_w is

$$\frac{p_w}{p_\infty} = 4j\left[\frac{\gamma JS^2}{3\gamma - 1}\left(\frac{\gamma - 1}{2}\right)^{\omega+1} \times \frac{\nu_\infty}{u_\infty x \ln(x^3 M_\infty^{2\omega}\nu_\infty/u_\infty r_w^4)}\right]^{1/2} M_\infty^{2+\omega} \quad (3.33)$$

$p_w/(p_w)_{E=0}$ is plotted in Fig. 2a for $\omega = 1$ and $Pr = 0.7$.

If the linear-viscosity approximation is used for pressure, the same proportionality constant C should be chosen as was chosen in formulas for C_f and q_w, since S^2 is independent of ω.

2. *Weak-shock limit*

If for large x the body thickness grows as x^n, $n < \frac{3}{4}$, neither the body nor the boundary layer grows sufficiently fast to sustain a strong shock wave, and p_w will approach p_∞ uniformly in r as $x \to \infty$. In this linearized limit

$$\delta^2 \propto \int_0^x JS^2 dx$$

If then $r_w/(SJ^{1/2}) \propto x^n$ where $n < \frac{1}{2}$, we are assured that $\delta/r_w \gg 1$ for sufficiently large x. In this case

$$\delta^2 = 4\left(\frac{\gamma-1}{2} M_\infty{}^2\right)^{\omega+1} \frac{\dfrac{\nu_\infty}{u_\infty}\displaystyle\int_0^x JS^2 dx}{\ln(xM_\infty{}^{2+2\omega}\,\nu_\infty/u_\infty r_w{}^2)} \qquad (3.34)$$

The surface pressure for large x is obtained from the linearized theory for supersonic potential flow¶ over a slender body of cross section $\pi\delta^2$. Such a flow can be generated by a line of sources on the axis ($r = 0$, $x > 0$). The disturbance potential ϕ where $M_\infty r/x \ll 1$ is given[12] by

$$\phi = -\frac{u_\infty}{2}\left[(\delta^2)_x \ln\frac{2x}{M_\infty r} + \int_0^x (\delta^2)_{\xi\xi} \ln(x-\xi)d\xi\right] \qquad (3.35)$$

Using the axisymmetric form of the surface pressure coefficient

$$\frac{p_w}{p_\infty} - 1 = \frac{\gamma}{2} M_\infty{}^2 \left[(\delta^2)_{xx} \ln\frac{2x}{M_\infty\delta} + \frac{d}{dx}\int_0^x (\delta^2)_{\xi\xi}\ln(x-\xi)d\xi + \frac{1}{x}(\delta^2)_x - (\delta_x)^2\right] \qquad (3.36)$$

If g_w and E are constant, (3.36) reduces to

$$\frac{p_w}{p_\infty} - 1 = \frac{3\gamma}{8}\left(\frac{M_\infty\delta}{x}\right)^2 = \frac{3\gamma}{2}\left(\frac{\gamma-1}{2}\right)^{\omega+1} \times \frac{JS^2\nu_\infty M_\infty{}^{2(2+\omega)}}{u_\infty x \ln(xM_\infty{}^{2+2\omega}\,\nu_\infty/u_\infty r_w{}^2)} \qquad (3.37)$$

$[(p_w/p_\infty) - 1]/[(p_w/p_\infty) - 1]_{E=0}$ is the square of the pressure ratio plotted in Fig. 2a.

IV. Slender $\frac{3}{4}$ Power-Law Bodies in Hypersonic Flow

Consider slender axisymmetric bodies of the form $r_w \propto x^{3/4}$ with strong shock waves and constant g_w and $(\rho v r)_w$. In this case the flow is self-similar. Numerical results have been obtained for cases where effects of transverse curvature and surface mass addition are small as well as large.

The self-similar equations have been integrated by marching from the wall, for $\beta = \frac{1}{14}$ (i.e., $\gamma = 1.4$), $Pr = 0.7$, $\omega = 1$, $g_w = 0$ and specified values of A and E. Results are given in Table 1. The skin friction, heat transfer, boundary-layer thickness, and pressure are then found from (2.13, 2.15, 3.30, and 3.31). Equations (2.5), with $\omega = 1$, (2.14), and (3.31) indicate

$$\frac{C_D}{(C_D)_{in}} = R(\infty)\left[1 + \frac{4}{3}\frac{\gamma}{\gamma-1} A f_{\eta\eta,w}\right] \qquad (4.1)$$

where $(C_D)_{in} = (3j^2/\gamma)(r_b/L)^2$ is the drag coefficient for inviscid flow over a $\frac{3}{4}$ power-law body referenced to base area.

Introduce

$$\Lambda \equiv \frac{M_\infty}{(r_w/x)^2}\left(\frac{C\nu_\infty}{u_\infty x}\right)^{1/2} = \frac{jA}{\gamma-1}[R(\infty)]^{1/2} \qquad (4.2)$$

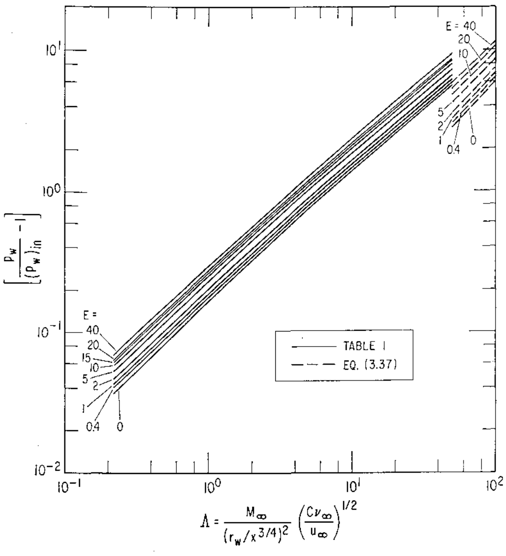
where $C \equiv (T_r/T_\infty)^{\omega-1}$ as previously noted. Since Λ is known in terms of freestream conditions and body geometry, it is convenient to consider the flow properties to be functions of Λ rather than A, as presented in Fig. 4. It is seen that ablation increases pressure and boundary-layer thickness and decreases skin friction and heat transfer. The net effect on total drag is that the drag decreases with increase in the ablation rate for any A (Fig. 4e).

For comparison, the asymptotic results for large Λ are also shown in Fig. 4, as dashed lines. These results, obtained by setting $\omega = 1$ in (3.22–3.24, 3.31, and 3.32), differ in absolute value from the exact numerical results too much for direct application of the asymptotic theory when $\ln\Lambda = O(1)$. Nevertheless, Fig. 4 shows that the *relative* effect of mass transfer given by the theory (through J, S^2, and λ) provides a useful bound on the relative effect of mass transfer when δ/r_w is finite. For further comparison, velocity profiles are shown in Fig. 4f for a representative value for E of 10.

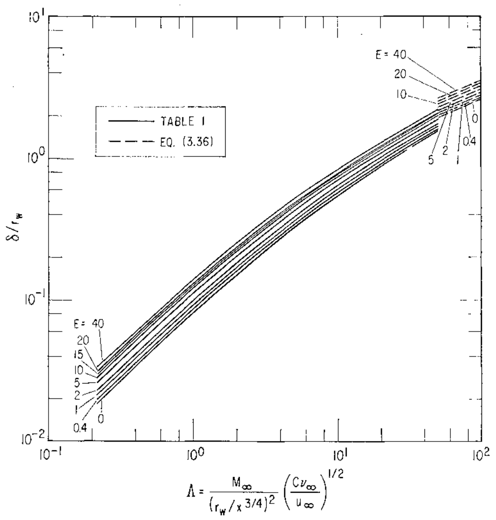
Table 1 Numerical solution of Eqs. (2.2–2.6) for self-similar flow ($Pr = 0.7$, $\beta = \frac{1}{14}$, $g_w = 0$, $\omega = 1$)

A	E	$f_{\eta\eta,w}$	$g_{\eta,w}$	$(h/H_\infty)_{\max}$	$[R(\infty) - 1]/A$
0	0.0	0.4871	0.3482	0.1865	0.3743
	0.4	0.4047	0.2923	0.1861	0.3983
	1.0	0.3238	0.2375	0.1857	0.4259
	2.0	0.2434	0.1829	0.1856	0.4598
	5.0	0.1384	0.1107	0.1861	0.5217
	10.0	0.07850	0.06800	0.1871	0.5759
	15.0	0.05361	0.04941	0.1881	0.608
	20.0	0.04012	0.03890	0.1886	0.630
	40.0	0.01895	0.02114	0.1902	0.677
1	0.0	0.5224	0.3735	0.1861	0.4034
	0.4	0.4356	0.3148	0.1855	0.4311
	1.0	0.3500	0.2570	0.1854	0.4630
	2.0	0.2643	0.1990	0.1854	0.5028
	5.0	0.1513	0.1217	0.1860	0.5764
	10.0	0.08610	0.07544	0.1875	0.6428
	15.0	0.05883	0.05510	0.1886	0.682
	20.0	0.04402	0.04353	0.1894	0.710
	40.0	0.02069	0.02384	0.1913	0.769
5	0.0	0.6510	0.4657	0.1851	0.5109
	0.4	0.5477	0.3965	0.1849	0.5499
	1.0	0.4443	0.3274	0.1848	0.5970
	2.0	0.3391	0.2570	0.1851	0.6568
	5.0	0.1972	0.1609	0.1865	0.7713
	10.0	0.1133	0.1018	0.1886	0.878
	15.0	0.07757	0.07520	0.1900	0.943
	20.0	0.05803	0.05988	0.1912	0.989
	40.0	0.02708	0.03335	0.1939	1.090
10	0.0	0.7935	0.5678	0.1847	0.6288
	0.4	0.6715	0.4866	0.1847	0.6809
	1.0	0.5482	0.4048	0.1846	0.7442
	2.0	0.4213	0.3205	0.1850	0.8255
	5.0	0.2475	0.2036	0.1868	0.9833
	10.0	0.1430	0.1304	0.1893	1.133
	15.0	0.09814	0.09697	0.1911	1.225
	20.0	0.07345	0.07757	0.1924	1.289
	40.0	0.03420	0.04362	0.1954	1.434
50	0.0	1.6840	1.2053	0.1835	1.368
	0.4	1.4434	1.0479	0.1836	1.499
	1.0	1.1945	0.8854	0.1839	1.661
	2.0	0.9318	0.7138	0.1848	1.873
	5.0	0.5600	0.4675	0.1876	2.296
	10.0	0.3283	0.3067	0.1909	2.706
100	0.0	2.5867	1.8508	0.1831	2.1166
	0.4	2.2257	1.6156	0.1833	2.329
	1.0	1.8498	1.3717	0.1837	2.591
	2.0	1.4498	1.1120	0.1847	2.937
	5.0	0.8778	0.7350	0.1877	3.630
	10.0	0.5172	0.4857	0.1912	4.307

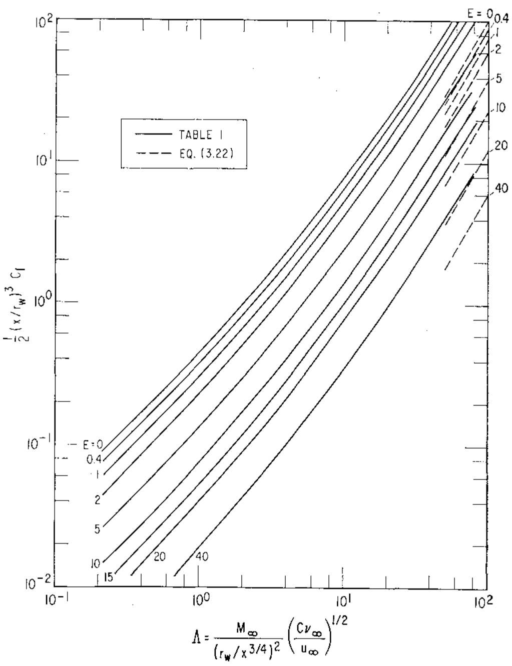
¶ The special case of a nonporous, isothermal body in the present limit of weak shock and large transverse curvature has been studied concurrently by Cross and Bush.[11] They confirm our statement that $p_r = 0$ in the rotational flow. (Their computation of transverse velocity component is questioned, but is unrelated to p_r.)



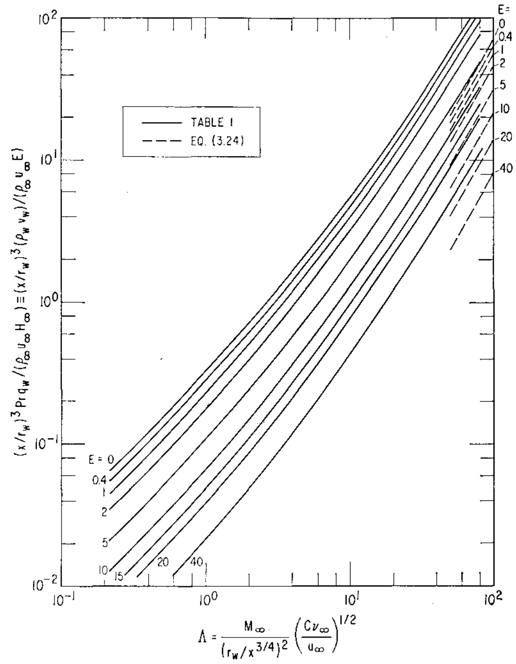
a) PRESSURE



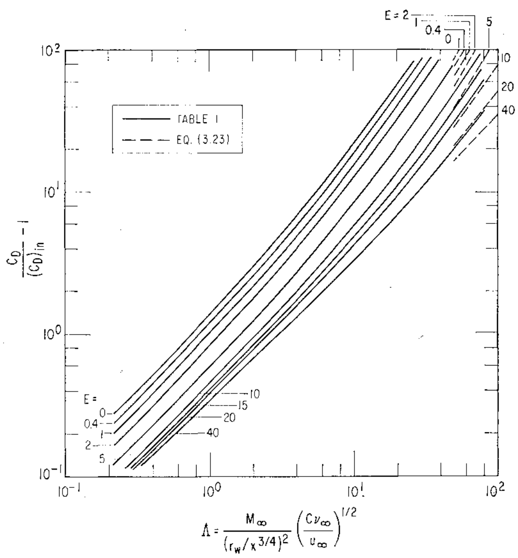
b) BOUNDARY LAYER THICKNESS



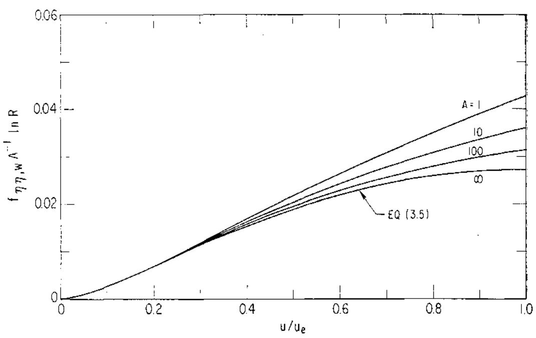
c) SKIN FRICTION



d) HEAT AND MASS TRANSFER



e) DRAG



f) VELOCITY PROFILES FOR E = 10

Fig. 4 Effects of mass transfer and transverse curvature on slender axisymmetric bodies in hypersonic flow; $r_w \sim x^{3/4}$, $Pr = 0.7$, $\gamma = 1.4$, $g_w = 0$.

These clearly approach the inner viscous layer solution as $A \to \infty$. Temperature profiles change similarly with A, but there is little excursion in the maximum enthalpy ratios, which for $E = 10$ are expected to approach 0.1832 as $A \to \infty$.

V. Concluding Remarks

A theory has been presented for boundary-layer development on a very slender body in hypersonic flow with arbitrary distributions of body thickness, surface temperature, and mass transfer. The boundary layer was found to consist of an outer layer, where viscous and convective effects are of comparable magnitudes, and an inner viscous layer, where the major property variations occur. The inner viscous layer equations were integrated analytically, and expressions for surface shear, heat transfer and net drag were obtained [Eqs. (3.22–3.24)]. These results require a knowledge of only the order of magnitude of the axial pressure distribution. An analytic expression relating boundary-layer thickness and pressure distribution was also obtained [Eq. (3.28)]. The streamwise variation of boundary-layer thickness and pressure was found for the case of a very strong shock ($p_w/p_\infty \gg 1$) and for a very weak shock ($p_w/p_\infty \to 1$). The foregoing results are independent of the profile structure in the viscous convective layer and therefore did not require a numerical integration of the equations describing this layer. Whereas the viscous convective layer can only be described in terms of a single similarity variable ζ if the body surface has uniform g_w, E, and β, it was necessary to invoke such similarity only once, to find p_w and δ when the shock is strong.

The gas was restricted by assuming uniform Pr, γ, and ω. For real gases with more than one species the theory should prove useful in predicting the relative effect of mass transfer, provided the molecular weights of the various species are not widely different.

With increasing mass injection, the present theory fails when $f_{\eta\eta}$ and g_η in the inner viscous layer are no longer large compared with their values in the viscous convective layer (which are of order $1/A$). Representative of $f_{\eta\eta}$ and g_η in the inner layer are their surface values, since it can be shown that for all allowable mass-transfer rates, the maximum shear stress occurs at the wall. Since for large mass injection $g_{\eta,w} > f_{\eta\eta,w}$, the more critical parameter is $f_{\eta\eta,w}$, and for the present theory to be valid we require $1 \ll Af_{\eta\eta,w}$, or

$$E \ll A^{2Pr} \propto \begin{cases} \left[\dfrac{x^2}{r_w^2} M_\infty{}^\omega \left(\dfrac{\nu_\infty}{u_\infty x}\right)^{1/2}\right]^{Pr}, & \dfrac{p_w}{p_\infty} \gg 1 \quad (5.1\text{a}) \\ \left[\dfrac{x^2}{r_w^2} M_\infty{}^{2+2\omega} \dfrac{\nu_\infty}{u_\infty x}\right]^{Pr}, & \dfrac{p_w}{p_\infty} - 1 \ll 1 \quad (5.1\text{b}) \end{cases}$$

Equations (5.1) were obtained without consideration of logarithmic orders. The latter must be retained if an upper limit on $(\rho vr)_w$ is desired. From (3.26), for large E, $[(\rho vr)_w/\mu_s] \ln A = \lambda_0 EJ \sim g_w{}^\omega \ln(\lambda_0 E)$. For $g_w \neq 0$, the requirement (5.1) becomes $(\rho vr/\mu)_w < 2$; i.e., the "wall Reynolds number" must be less than 2. The case of strong mass injection with large transverse curvature, $E = O(A^{2Pr}) \gg 1$, has not been investigated. Limitations of the theory due to other assumptions are the same as for the theory without mass transfer.[3] For an ablating vehicle with uniform Q^*, if the body is initially a slender cone, it becomes and remains a hyperboloid of increasing nose radius.

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